The Inverse Transform

A Different Transform In Name Only

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Abstract

This paper proves the inverse form of the Lorentz Transformation is mathematically identical to the primary form, and that it is simply the primary form expressed in a different format.

1 Introduction

The Lorentz Transformation pertains to two inertial observers. The primary form of the transformation is generally used to transform the first observer's spacetime coordinates of an event into the second observer's spacetime coordinates for the same event. The inverse form of the transformation is generally used in the opposite way, to transform the second observer's spacetime coordinates of an event into the first observer's spacetime coordinates for the same event. However, the reader should understand that the primary and inverse forms of the transformation are mathematically indistinguishable. To be clear, they are mathematically identical. They simply express the relationship of the first and second observer's spacetime representation of the same event in two different ways. It has been over a hundred years since the transformations were placed in the limelight, and yet even today, many people still believe there is a fundamental difference between the primary and inverse form of the transformation. I think I understand why since I was one of those people. I became interested in physics two years ago, and one of the first things I did was derive the primary and inverse form of the transformation which are presented below in Sections 1 and 2. The two sets of transformation equations looked considerably different to me, and I erroneously concluded they were different. But, they're not! You can use whichever set you want to use; you'll get the same answer with either set. However the amount of mathematics involved can be different using one set versus the other set. I'm pretty sure that there are some math gurus who can intuitively grasp the equivalence of the two forms of the transformation, but for the rest of us we need step by step proof to believe it. I provide that proof in Section 4. I derive the **INVERSE** form of the transformation from the **PRIMARY** form thereby proving the two forms are identical. In all of the derivations that follow I apply a $(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t})$ spacetime coordinate system to the first observer who I call Jane, and a (x', y', z', t') spacetime coordinate system to the second observer who I call Dick. For more information about Dick and Jane consult my book **[1]** on relativity which is available on Amazon.

2 The Primary Transformation

The primary form of the *Lorentz Transformation* is shown below. The equations were derived assuming Jane to be at rest in space and Dick to be in motion relative to Jane at velocity $\boldsymbol{v}$ in the positive ***xx'*** direction.

$$x' = \gamma\,(x - vt) \tag{2.1}$$
$$y' = y \tag{2.2}$$
$$z' = z \tag{2.3}$$
$$t' = \gamma(t - xv/c^2) \tag{2.4}$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.5}$$

3 The Inverse Transformation

The inverse form of the *Lorentz Transformation* is shown below. The equations were derived assuming Dick to be at rest in space, and Jane to be in motion relative to Dick at velocity $\boldsymbol{v}$ in the negative ***xx'*** direction.

$$x = \gamma\,(x' + vt) \tag{3.1}$$
$$y = y' \tag{3.2}$$
$$z = z' \tag{3.3}$$
$$t = \gamma(t' + xv/c^2) \tag{3.4}$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3.5}$$

4 Deriving The Inverse From The Primary

The primary form of the transformation shown in Section 2, expresses each of Dick's four spacetime coordinates (primed variables) in terms of combinations of Jane's four spacetime coordinates (un-primed variables). The inverse form of the transformation shown in Section 3 is formatted in the oppsite way. Each of Jane's four spacetime coordinates (un-primed variables) are expressed in terms of combinations of Dick's four spacetime coordinates (primed variables). The following steps show that the inverse form can be derived from the primary form by simple algebra. The step by step derivation to show this equivalence begins with the first equation in Section 2 as shown below.

Expand right side of equation (2.1) then rearrange and solve for $\boldsymbol{x}$ as shown below.

$$x' = \gamma x - \gamma v t \tag{4.1}$$
$$\gamma x = x' + \gamma v t \tag{4.2}$$
$$x = x'/\gamma + vt \tag{4.3}$$

Expand right side of equation (2.4) and solve for t as shown below.

$$t' = \gamma t - \gamma x v/c^2 \tag{4.4}$$
$$t = t'/\gamma + xv/c^2 \tag{4.5}$$

Substitute t from equation (4.5) into equation (4.3) as shown below.

$$\begin{aligned} x &= x'/\gamma + v(t'/\gamma + xv/c^2) \\ &= x'/\gamma + v\,t'/\gamma + xv^2/c^2 \end{aligned} \tag{4.6}$$

Rearrange equation (4.6) so primes are on the right.

$$x(1 - v^2/c^2) = x'/\gamma + v\,t'/\gamma \tag{4.7}$$

Apply identity $(1 - v^2/c^2) = 1/\gamma^2$ to equation (4.7) then solve for x.

$$x/\gamma^2 = x'/\gamma + v\,t'/\gamma \tag{4.8}$$
$$x = \gamma(x' + v\,t') \tag{4.9}$$

The inverse of equations (2.2 and 2.3) is a simple left-right reversal as shown below.

$$y = y' \tag{4.10}$$
$$z = z' \tag{4.11}$$

Substitute $\boldsymbol{x}$ from equation (4.9) into equation (4.5) and then simplify the expression for $\boldsymbol{t}$ as shown below.

$$\begin{aligned} t &= t'/\gamma + xv/c^2 \\ &= t'/\gamma + \gamma(x' + v\,t')v/c^2 \\ &= t'/\gamma + \gamma x'v/c^2 + \gamma\,t'v^2/c^2 \\ &= \gamma t'(1/\gamma^2 + v^2/c^2) + \gamma\,x'v/c^2 \quad \textit{Identity} \quad 1/\gamma^2 = 1 - v^2/c^2 \\ &= \gamma t'(1 - v^2/c^2 + v^2/c^2) + \gamma\,x'v/c^2 \\ &= \gamma t' + \gamma\,x'v/c^2 \\ &= \gamma(t' + x'v/c^2) \end{aligned} \tag{4.12}$$

The γ factor in eqquation (2.5) is not a function of Dick and Jane's spacetime coordinates, and is the same in both forms of the transformation.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{4.13}$$

5 Summary

The concordance of the derivation in Section 4 with the inverse form of the transformation in Section 3 is shown below.

- Equation (3.1) = Equation (4.9)
- Equation (3.2) = Equation (4.10)
- Equation (3.3) = Equation (4.11)
- Equation (3.4) = Equation (4.12)
- Equation (3.5) = Equation (4.13)

The above concordance proves that the two forms of the transformation are identical, and yet I erroneously concluded the two forms were different when I first derived them. My error in logic stemmed from the assumptions I made in deriving the two forms of the transformation. I was trying to mimic the way I envisioned Lorentz would have derived his equations. The fact that I assumed Jane to be at rest and Dick in motion for the first derivation, and the opposite assumption for the second derivation caused me to subconsciously infer that this distinction in the different assumptions would impart a corresponding distinction in the two forms of the transformation. This is of course an illogical inference. The correct inference is that the transformations must be compatible with the conditions in the assumptions used to derive them, not that they must mandate the conditions in the assumptions. For those of you interested in the logic path leading to the *Lorentz Transformation* check out my book [**1**] about Einstein's throry of relativity.

References

[**1**] Richard Alan, Everything You Ever Wanted To Know About Dick & Jane And Mary. "The derivation of Einstein's Special Theory of Relativity in nauseating detail". Available on Amazon.