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Tutorial

The Wave Equations

For Electromagnetic Waves

Abstract

This tutorial provides a supplemental explanation about the derivation presented in the referenced paper and about the undefined shape function $f(x)$ that was a key element in the derivation. This tutorial is not intended to be a substitute for the paper. In fact, there is so much digression in this tutorial that it detracts from presenting a clear logic path to the conclusion in the referenced paper. It should only be used in conjunction with the referenced paper as an aid in following its logic path.

Eq.1 and Eq.2

The derivation presented in the referenced paper was based solely on a rigid plastic model's depiction of fields rather than on scientific theories or experimental results involving electricity and magnetism. The purpose in constraining the derivation in this way was to provide clear justification for asserting that the wave equations resulting from the derivation pertained to a rigid physical entity propagating through space. The plastic model used in the derivation consisted of a long black plastic arrow with two orthogonally positioned plastic fins attached. One fin was shaded blue to denote the electric field; the other red to denote the magnetic field. The arrow depicted the actual wave; it represented a single strand of electromagnetic energy and it denoted the direction the plastic model would move when its velocity was greater than zero. The amplitude of the fins at each point along the arrow represented the strength of the fields within the strand of energy represented by the arrow. A shape function $f(x)$ specified the amplitude of the fins mathematically for each point x along the arrow. In essence, this shape function, which was introduced in Eq.1 and Eq.2 of the referenced paper, formed the basis of the mathematical derivation of the wave equations. A copy of Eq.1 and Eq.2 from the referenced paper is shown below.

1. $A_y = k_y \quad f(x)$	Blue field strength (zero velocity)
2. $A_z = k_z \quad f(x)$	Red field strength (zero velocity)

The bottom line. Think of fin amplitude as the number of inches the fin protrudes perpendicular to the black arrow in the model. And, think of the function $f(x)$ as specifying this amplitude based on the value of its argument x. Even though we haven't defined the function $f(x)$, we can assume a mathematical expression defining the function could be created for any wave shape we can imagine, and that it would denote the fin amplitude for every point x along the arrow. Once we accept that assumption, we can accept the mathematical depiction of field strength shown in Eq.1 and Eq.2. These two equations express the field strength A_y and A_z as simply being proportional to the shape and size of the plastic fins denoted by the shape function $f(x)$ even though the function has not been defined. In order to clarify why the shape function $f(x)$ was introduced we need to clarify what a function like $f(x)$ is, and what some of its mathematical attributes are. The function $f(x)$ is simply a shorthand notation for a mathematical expression that is based on the argument of the function. For example, the shape function could be defined to represent a simple expression like in the following statement.

let $f(a) = 4\,a$ Simple shape function

This definition simply asserts that the value of $f(a)$ is 4 times the value of the argument a. The following three examples demonstrate the evaluation of $f(a)$ as defined above. In each example the argument a, which could be assigned any value, is arbitrarily assigned three different values.

$f(2)$	$= 4 * 2$	$= 8$	Three example evaluations of
$f(2+4)$	$= 4 * (2+4)$	$= 24$	the simple shape function defined
$f(p+q)$	$= 4 * (p+q)$	$= 4p + 4q$	above.

There is only a single argument for the shape function in each of the three examples. You might think that in the last example there were two arguments. There were not. There was simply one argument containing two terms. This is not uncommon. The arguments of a function often contain multiple terms or factors. This last example is important because that form of the shape function argument appears repeatedly in the derivation presented in the referenced paper. It is repeated below so we can study it in more detail.

$$f(p+q) = 4 * (p+q) = 4p + 4q$$

If we assume p and q are independent variables then $dq/dp = dp/dq = 0$. Under this assumption the derivatives with respect to p and with respect to q are,

$$df(p+q)/dp = 4 + 4\,dq/dp = 4$$
$$df(p+q)/dq = 4 + 4\,dp/dq = 4$$

Note that the derivatives are equal. This should have been intuitively obvious but having it asserted mathematically bolsters the intuition. Note that the derivatives would be equal even if the multiplicative constant in the definition of the function would have been 5 instead of 4. In fact, they would have been equal even if the multiplicative constant would have been some god-awful expression to complex to write. More importantly, they would have been equal even if the function had never been defined. The reason for this is that p and q are assumed to be independent. Note that if the variables in an expression are known to be independent then you'll often see partial derivative notation rather than the derivative notation shown above. The equality of the derivatives of the shape function above can be expressed in partial derivative notation as,

$$\partial f(p+q)/\partial p = \partial f(p+q)/\partial q \qquad \text{Derivative with respect to } p \text{ and } q$$

The partial derivative notation above carries with it the implied assumption that p and q are independent. This notation is used exclusively in the referenced paper.

<u>Eq.3 and Eq.4</u>

Eq.1 and Eq.2 apply to the plastic model when it is stationary. Beginning with Eq.3 and Eq.4 the plastic model is assumed to be moving at the constant velocity v in the positive X direction. When the plastic model is in motion the argument for the shape function requires two terms x and $-vt$. This was shown in Eq.3 and Eq.4. of the referenced paper. A copy of the two equations is shown below.

3. $A_y = k_y \; f(x - vt)$	Blue field strength (velocity v)
4. $A_z = k_z \; f(x - vt)$	Red field strength (velocity v)

It is easy to understand the reason for the two terms in the argument of the shape function. Imagine the observer viewing the <u>stationary</u> plastic model at the point on the black arrow at x=0. Now imagine he holds his viewpoint fixed as his assistant pulls the model in the positive X direction at velocity v for t seconds. The plastic model will have moved a distance of vt and is now <u>stationary</u> again. The observer would now be viewing a point on the black arrow that he would have seen before the move at $x = -vt$. With the model at this new position the amplitude of the fins at the observers viewpoint $x = 0$ would now be $f(-vt)$. If the observer shifts his viewpoint along the arrow to any point other than $x = 0$ then the amplitude of the fins would be equal to $f(x\text{-}vt)$. Therefore, this is the appropriate expression for the shape function argument when the plastic model is moving at constant velocity v in the positive X direction.

Eq.5 and Eq.6

Note that in the previous section the variables *x* and *t* were independent. Recall that if the argument of the shape function is the sum of two independent variables then the derivative of the function with respect to the first term is equal to the derivative with respect to the second term. This equivalence was expressed in Eq.5 of the referenced paper using partial derivative notation. It is repeated below.

5. $\partial f(x - vt) / \partial x \;=\; \partial f(x - vt) / \partial(-vt)$	Derivative with respect to *x* and -*vt*

Since *v* is constant, the factor -*v* in the denominator on the right side of the above equation can instead be expressed as a multiplicative factor of the derivative as was shown in Eq.6 in the referenced paper. A copy of that equation is shown below.

6. $\partial f(x - vt) / \partial x \;=\; (-1/v)\; \partial f(x - vt) / \partial t$	Derivative with respect to *x* and *t*

The above equation is interesting. The left side shows the magnitude of a spatial derivative as being equal to a time derivative shown on the right side. A student of Maxwell might recall that his two equations that apply to waves in space equate curl on the left side with a time derivative on the right side. This is kind of like Eq.6 but the similarity is weak. To get a glimpse of where we are headed we need to jump ahead to Maxwell's equations.

Eq.18 and Eq.19

The two Maxwell's equations that apply to waves in space are vector equations relating the curl of one variable to the time derivative of a second variable. When those vector equations are expressed in partial derivative notation they contain many terms and are very complex. However, when those equations are applied to a wave in which the XYZ coordinate has been carefully aligned to the polarization of the wave as depicted in the referenced paper, then they become greatly simplified (Satterfield[1]). Instead of two vector equations containing many terms you end up with two scalar equations containing only a couple of terms. They are shown in the referenced paper as Eq.18 and Eq.19 and are repeated below.

18. $\partial E_y / \partial x \;=\; -\mu_0 \;\; \partial H_z / \partial t$	MAXWELL'S EQUATIONS
19. $\partial H_z / \partial x \;=\; -\varepsilon_0 \;\; \partial E_y / \partial t$	(for wave depicted in drawing)

It's easy to see where we're headed when we look at the above equations. We are looking for an expression where the spatial derivative of the red fin is proportional to the time derivative of the blue fin, and vice versa. With that in mind we go to Eq.7 ... Eq.10 in the referenced paper. A copy of those equations are shown below.

7. $\partial A_y / \partial x \;=\; k_y \; \partial f(x - vt) / \partial x$
8. $\partial A_z / \partial x \;=\; k_z \; \partial f(x - vt) / \partial x$
9. $\partial A_y / \partial t \;=\; k_y \; \partial f(x - vt) / \partial t$
10. $\partial A_z / \partial t \;=\; k_z \; \partial f(x - vt) / \partial t$

There is nothing magic in those equations. They are simply the spatial derivative and the time derivative of Eq.3 and Eq.4. The only thing mysterious about expressing those four equations is why we might want them. The answer is obvious. We intend to show that they can be mathematically transformed into a form like Eq.18 and Eq.19, and this is where Eq.6 comes into play.

If we substitute the right side of Eq.6 for the spatial derivatives on the right side of Eq.7 and Eq.8 we get ,

11. $\partial A_y / \partial x = -k_y / v \;\; \partial f(x - vt) / \partial t$
12. $\partial A_z / \partial x = -k_z / v \;\; \partial f(x - vt) / \partial t$

We are getting close but the right side of Eq.11 and Eq.12 is not quite right; we want the right side to only involve the terms A_y and A_z. Therefore we substitute from the left side of Eq.9 and Eq.10 to the right side of Eq.11 and Eq.12 we get,

13. $\partial A_y / \partial x = -1 / v \;\; \partial A_y / \partial t$
14. $\partial A_z / \partial x = -1 / v \;\; \partial A_z / \partial t$

Now we are almost there. From Eq.3 we can solve for,

$$A_y / k_y = f(x - vt)$$

and similarly from Eq.4 we get,

$$A_z / k_z = f(x - vt)$$

Equating the left side of the above equations we get,

$$A_y / k_y = A_z / k_z$$

Rearranging terms we can express the proportionality of A_y and A_z as shown in Eq. 15 of the referenced paper.

15. $k_z \; A_y = k_y \; A_z$

The proportionality expressed in Eq.15 allows us to rewrite the right side of Eq.13 and Eq.14 to get,

16. $\partial A_y / \partial x = -k_y /(k_z v) \;\; \partial A_z / \partial t$ BIPOLAR TRANSVERSE WAVE EQUATIONS
17. $\partial A_z / \partial x = -k_z /(k_y v) \;\; \partial A_y / \partial t$ (for field strength proportional to fin amplitude)

Voilà ! Eq.16 and Eq.17 are exactly the same form as Maxwell's equations shown in Eq.18 and Eq.19. The only thing left is to solve for the values of the three parameters k_y, k_z, and v that make the Bipolar Transverse Wave Equations identical to Maxwell's equations. This was done elegantly in the referenced paper so there is no need to repeat it here.

Purpose

The purpose of this tutorial is to refresh the reader's understanding of complicated mathematical manipulations in the referenced paper so that he will not be distracted by puzzling math steps. It is NOT intended that this tutorial be a substitute for the paper. In fact, after studying this tutorial the reader should be able to follow the logic path in the referenced paper without referring to the tutorial.

References

1. *The Birth of a Wave. Satterfield, www.richard-alan.com* Section 3 derives Maxwell's equations applied to a wave in which the XYZ coordinate has been carefully aligned to the polarization of the wave.